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A study on an analytic solution 1D heat equation of a parabolic partial differential equation and implement in computer programming

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Abstract— In this paper, we investigate second order parabolic partial differential equation of a 1D heat equation. In this paper, we discuss the derivation of heat equation, analytical solution uses by separation of variables, Fourier Transform and Laplace Transform. Finally, we consider a problem of heat equation and the solution of this problem implement in computer programming.

Index Terms— Heat equation, Fourier law, Fourier Transformation, Laplace Transformation, Analytical solution, Separation of variable, Mat-lab.

1 History of Heat Equation

The heat equation is an important partial differential equation (PDE) which describes the distribution of heat (or variation in temperature) in a given region over time. For better understanding of this paper, it is very important that we understand the difference between heat and temperature. Heat is a process of energy transfer as a result of temperature difference between the two points. Thus, the term 'heat' is used to describe the energy transferred through the heating process. Temperature, on the other hand, is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore, no heat would be exchanged between bodies of the same temperature.

Suppose we have a function $(x;\ y;\ z;\ t)$, which describes the temperature of a conducting material at a given location, $(x;\ y;\ z)$, you can use this function to determine the temperature at any position on the material at a future time, $t+1$. The function U changes over time as heat spreads through-out the material and the heat equation is used to determine this change in the function U. The gradient of U describes which direction and at what rate is the temperature changing around a particular region of the material.

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Therefore, the gradient of temperature is the ow of heat through the material. This gradient will help us determine the ow of heat through various materials. This is analogous to the ow of water in a pipe.

The heat equation is a parabolic partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time. For a function (x, y, z, t) of three spatial variables (x, y, z) (see Cartesian coordinates) and the time variable t, the heat equation is:

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

More generally in any coordinate system:

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

Where α is a positive constant, and Δ *or* ∇^2 denotes the Laplace operator. In the physical problem of temperature variation, $u(x, y, z, t)$ is the temperature and α is the thermal diffusivity. For the mathematical treatment it is sufficient to consider the case $\alpha = 1$.

Note that the state equation, given by the first law of thermodynamics (i.e. conservation of energy), is written in the following form (assuming no mass transfer or radiation). This form is more general and particularly useful to recognize which property (e.g. c_p or ρ) influences which term.

$$\rho c_p \frac{\partial T}{\partial t} - \nabla.(k\nabla T) = \dot{q}v$$

Where $\dot{q}v$ is volumetric heat flux.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics it is the prototypical parabolic partial differential equation. In probability theory, the heat equation is connected with the study of Brownian motion via the Fokker-Planck equation. In financial mathematics it is use to solve the Black-Scholes partial differential equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

The heat equation is used in probability and describes random walks. It is also applied in financial mathematics for this reason.

It is also important in Riemannian geometry and thus topology: it was adapted by Richard S. Hamilton when he defined the Ricci flow that was later used by Grigori Perelman to solve the topological Poincare conjecture.

The aim of this paper is to be able to determine the ow of heat of various materials i.e. different thermal conductivities. Does the arrangement of conductors or insulators affect the rate at which the heat owes? Imagine a room with a wall that is made of different materials such as wood, metal or bricks arranged in different ways. The room is at room temperature, say $25^{o}C$ and does not generate any heat (no air conditioner) and it is surrounded by the outside environment which has a temperature of $0^{o}C$. The room is so tiny relative to the outside environment therefore any heat ow from the room to the outside would not change the temperature outside. However, the temperature inside the room is prone to changes due to the surrounding temperature. How can we ensure that we maintain the room temperature for the longest possible time without the use of an air conditioner? If the walls of the room are bad insulators of heat, it is almost impossible to maintain the room temperature. This is when it is important that we maximize the materials and knowledge that we have to build a wall that would keep the room temperature constant. It is possible that one can just buy building materials with low thermal conductivity. However, the constraints are that we have a variety of bad and good thermal conductors and we are trying to build the best congruation with the materials that we have.

To answer these questions, we have created materials with different thermal conductivities arranged different ways. We are more interested in two cases:

I. what happens to the heat ow when we reverse the order of thermal conductivities and
II. what happens when we put the materials with high thermal conductivities on the edges or vice versa.

To test these arrangements, we will set the temperature on one end of the material to be at $0^{o}C$ and the other end at $100^{o}C$. But before we get into that, let us have a look at the two kinds of conduction that are important to the understanding of this paper.

To be able to solve the second-order partial differential heat equation in the spatial coordinates, we need to know the boundary conditions and the initial conditions. The boundary conditions specify how our system interacts with the outside surroundings. There are three general types of boundary conditions: Dirichlet, Neumann and Mixed boundary conditions.

The heat equation in one dimension is written as the following:

$$\frac{\partial U}{\partial t} = c\left(\frac{\partial^2 U}{\partial x^2}\right)$$

Where $U(x,t)$ is a function of temperature.

In this case we can think of a one-dimensional rectangular thin wire with length x. Ignore the width and height dimensionality. The one end of the length of the wire is set at 0o C whereas the other end is set at $0^{o}C$. These are its boundary conditions. We also need to specify the temperature at every position on the wire at time, t_o (the initial conditions). To solve this one-dimensional heat problem, we need to transform the above heat equation into an explicit method using the second-order central Finite Difference Method. Therefore, explicitly we can write the one-dimensional heat equation as:

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \alpha\left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}\right)$$

This equation can then be implemented and solved easily using Mat lab or other languages. The value should be much smaller than 1, other-wise you get unexpected errors. In this case, we assume that is a constant although later we are going to see that alpha could be de need as a function that depends on space. As we can see, the heat equation in 1-D explicit form is straight forward because the right side has only one term.

In a metal rod with non-uniform temperature, heat (thermal energy) is transferred from regions of higher temperature to regions of lower temperature. Three physical principles are used here.

I. **Heat (or thermal) energy of a body with uniform properties:** Heat $energy = cmu$, where m is the body mass, u is the temperature, c is the specific heat, units $[c] = L^2T - 2U - 1$ (basic units are M mass, L length, T time, U temperature). c is the energy required to raise a unit mass of the substance 1 unit in temperature.

II. **Fourier's law of heat transfer:** rate of heat transfers proportional to negative temperature gradient, Rate of heat transfer $\partial u = -K_o$ (1) area ∂x where K_o is the thermal conductivity, units $[K_o] = MLT - 3U - 1$. In other words, heat is transferred from areas of high temp to low temp.

III. **Conservation of energy:** Consider a uniform rod of length l with non-uniform temperature lying on the x-axis from $x = 0\ to\ x = l$. By uniform rod, we mean the density ϱ, specific heat c, thermal conductivity K_o, cross-sectional area A are all constant. Assume the sides of the rod are insulated and only the ends may be exposed. Also assume there is no heat source within the rod.

2 Derivation of Heat Equation

2.1 From Fourier's Law

The heat equation is derived from Fourier's law and conservation of energy (Cannon 1984). By Fourier's law, the rate of flow of heat energy per unit area through a surface is proportional to the negative temperature gradient across the surface,

$$q = -k\nabla u$$

Where k is the thermal conductivity and u is the temperature. In one dimension, the gradient is an ordinary spatial derivative, and so Fourier's law is

$$q = -k\frac{\partial u}{\partial x}$$

In the absence of work done, a change in internal energy per unit volume in the material, ΔQ, is proportional to the change in temperature, Δu (in this section only, Δ is the ordinary difference operator with respect to time, not the Laplacian with respect to space). That is, $\Delta Q = c_p\rho\Delta u$. Where cp is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at absolute zero temperature, this can be rewritten as $\Delta Q = c_p\rho u$.

The increase in internal energy in a small spatial region of the material $x - \Delta x \le \xi \le x + \Delta x$ over the time period $t - \Delta t \le \tau \le t + \Delta t$ is given by,

$$c_p\rho\int_{x-\Delta x}^{x+\Delta x}[u(\xi t + \Delta t) - u(\xi t + \Delta t)]d\xi$$

$$= c_p\rho\int_{t-\Delta t}^{t+\Delta t}\int_{x-\Delta x}^{x+\Delta x}\frac{\partial u}{\partial \tau}d\xi d\tau$$

where the fundamental theorem of calculus was used. If no work is done and there are neither heat sources nor sinks, the change in internal energy in the interval $[x - \Delta x, x + \Delta x]$ is accounted for entirely by the flux of heat across the boundaries. By Fourier's law, this is

$$k\int_{t-\Delta t}^{t+\Delta t}\left[\frac{\partial u}{\partial t}(x + \Delta x, \tau) - \frac{\partial u}{\partial t}(x - \Delta x, \tau)\right]d\tau$$

$$= k\int_{t-\Delta t}^{t+\Delta t}\int_{x-\Delta x}^{x+\Delta x}\frac{\partial^2 u}{\partial \xi^2}d\xi d\tau$$

Again, by the fundamental theorem of calculus. By conservation of energy,

$$\int_{t-\Delta t}^{t+\Delta t}\int_{x-\Delta x}^{x+\Delta x}\left[c_p\rho u_\tau - k u_{\xi\xi}\right]d\xi d\tau = 0$$

This is true for any rectangle $[t - \Delta t, t + \Delta t] \times [x - \Delta x, x + \Delta x]$. By the fundamental lemma of the calculus of variations, the integrand must vanish identically:

$$c_p\rho u_t - k u_{xx} = 0$$

Which can be rewritten as:

$$u_t = \frac{k}{c_p\rho}u_{xx} \quad or \quad \frac{\partial u}{\partial t} = \frac{k}{c_p\rho}\frac{\partial^2 u}{\partial x^2}$$

Which is the heat equation, where the coefficient (often denoted α), $\alpha = k/c_p\rho$ is called the thermal diffusivity.

An additional term may be introduced into the equation to account for radiative loss of heat, which depends upon the excess temperature $u = T - T_s$ at a given point compared with the surroundings. At low excess temperatures, the radiative loss is approximately μu, giving a one-dimensional heat-transfer equation of the form

$$\frac{\partial u}{\partial t} = \frac{k}{c_p\rho}\frac{\partial^2 u}{\partial x^2} - \mu u$$

At high excess temperatures, however, the Stefan–Boltzmann law gives a net radiative heat-loss proportional to $T^4 - T_s^4$, and the above equation is inaccurate. For large excess temperatures, $T^4 - T_s^4 \approx u^4$ giving a high-temperature heat-transfer equation of the form

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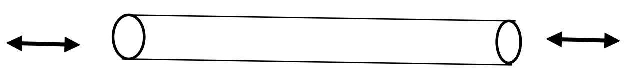
$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2}\right) - mu^4$$

Where $m = \epsilon\sigma p/\rho A c_p$. Here, σ is Stefan's constant, ε is a characteristic constant of the material, p is the sectional perimeter of the bar and A is its cross-sectional area. However, using T instead of u gives a better approximation in this case.

2.2 Using a Rod Pipe

Heat is the energy transferred from one body to another due to a difference in temperature. (Better: heat is the kinetic energy of the molecules that compose the material. Consider a long uniform tube surround by an insulating material like stir form along its length, so that heat can flow in and out only from its two ends:



There are two basic physical principle governing the motion of heat.

a) The total heat energy H contained in a uniform, homogeneous body is related to its temperature T and mass in the following simple way

$$H = k_s MT$$

Where k_s is the specific heat capacity of the material (a measurable constant specific to the material from with the body is made). More generally, in a situation for which neither the temperature nor the density of the material is constant we have

$$H(t) = k_s \int_v \rho(x)T(x,t)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

b) The rate of heat transfer across a portion S of the boundary of a region R of the body is proportional directional derivative of T across the boundary and the area of contact

$$Heat\ flux\ across\ S = \sigma \int_S \nabla T.ndS \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$$

Where n=n(x) is the direction normal to the surface of contact at the point x, and σ is another constant specific to the material from with the body is constructed. $\sigma$is called heat conductivity constant.

Applying Gauss's divergence theorem to (ii) we have

$$Heat\ flux\ entering\ or\ leaving\ a\ region = \sigma \int_{\partial R} \nabla T.ndS$$

$$= \sigma \int_R \nabla.\nabla T dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(iii)$$

This should be the total rate at which heat enters or leaves the region R, which in turn should correspond to the rate of change of the total amount of heat energy contained in the region:

$$\frac{dH}{dT} = k_s \int_R \rho\frac{\partial T}{\partial t}dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(iv)$$

Equating (iii) and (iv) we obtain

$$\sigma\int_R \nabla.\nabla T dx = k_s \int_R \rho\frac{\partial T}{\partial t}dx$$

Since the region R can be chosen arbitrarily, the two integrands must coincide at every point of the body. We thus obtain the heat equation

$$\nabla^2 T - \frac{\rho k_s}{\sigma}\frac{\partial T}{\partial t} = 0$$

In such a situation we can assume that the temperature really only6 depends on the position x along the length of the heat pipe. Then

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \approx \frac{\partial^2 T}{\partial x^2}$$

And the heat equation reduces to a 2-dimensional PDE of the form

$$\frac{\partial T}{\partial t} - \alpha^2\frac{\partial^2 T}{\partial x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(v)$$

$$Where, \quad \alpha = \sqrt{\frac{\rho k_s}{\sigma}}$$

(Replacing the ratio $\frac{\sigma}{\rho k_s}$ by α^2 will prove convenient later on.)

Or

We will now derive the heat equation with an external source,

$$u_t = \alpha^2 u_{xx} + F(x,t), 0 < x < L, t > 0,$$

where u is the temperature in a rod of length L, α^2 is a diffusion coefficient, and $F(x,t)$ represents an external heat source. We begin with the following assumptions:

- The rod is made of a homogeneous material.
- The rod is laterally insulated, so that heat flows only in the x-direction.
- The rod is sufficiently thin so that the temperature within any particular cross-section is constant.

These last two assumptions are used to allow us to treat the problem as one-dimensional. As we will see, the first assumption is not absolutely necessary, but it does simplify certain solution techniques. From the principle of conservation of energy, it follows that the heat within a segment of the rod $[x, x+\Delta x]$ satisfies the following:

Net change inside $[x, x+\Delta x]$ = Net inward flux across boundaries + Total heat generated inside $[x, x+\Delta x]$

The total amount of heat, in calories, in any segment $[a,b]$ is given by

$$\int_a^b c\rho A u(s,t)ds$$

where c is the thermal capacity of the rod (also known as the specific heat), ϱ is the density of the rod, and A is the cross-sectional area of the rod. In view of our assumptions, c, ϱ and A are constants. Also, recall that the flux from left to right at x = a is given by $-ku_x(a,t)$, where k is the thermal conductivity of the rod. Putting all of these facts together, we can translate the conservation relation into the equation

$$c\rho A\int_x^{x+\Delta x} u_t(s,t)\,ds = kA[u_x(x+\Delta x,t)-u_x(x,t)] + A\int_x^{x+\Delta x} f(s,t)\,ds$$

where $f(x,t)$ is the amount of heat generated by the external source per unit of length per unit of time. Note that we must use inward flux, which is why the flux term at $x = L$ must be negated. Applying the Fundamental Theorem of Calculus "in reverse",

$$f(b)-f(a)=\int_a^b f'(s)\,ds$$

we obtain, after dividing both sides by A,

$$c\rho\int_x^{x+\Delta x} u_t(s,t)\,ds = \int_x^{x+\Delta x} ku_{xx}(s,t)+f(s,t)\,ds$$

Rearranging yields

$$\int_x^{x+\Delta x} u_t(s,t)\,ds - \alpha^2 u_{xx}(s,t) - F(s,t)ds = 0$$

where, $\alpha^2=\frac{k}{c\rho}$, $F(x,t)=\frac{1}{c\rho}f(x,t)$ are the diffusivity of the rod and the heat source density respectively.

Since this equation holds on an arbitrary segment of the rod, it follows that the integrand must vanish everywhere in the rod, which yields the equation

$$u_t = \alpha^2 u_{xx} + F(x,t)$$

It is worth noting that the diffusivity $\alpha^2 = k/c\rho$ is proportional to the conductivity, but inversely proportional to the thermal capacity and the density. Physically, this makes sense because the more an object tends to store heat, and the denser it is, the more difficult it should be for heat energy to diffuse through the object, whereas the better the ability of the material to conduct heat, the easier it should be for heat energy to move through the object and diffuse.

2.3 Heat equation properties

We would like to solve the heat (diffusion) equation, $u_t - k\Delta u = 0$. And obtain a solution formula depending on the given initial data, similar to the case of the wave equation. However, the methods that we used to arrive at d'Alambert's solution for the wave IVP do not yield much for the heat equation. To see this, recall that the heat equation is of parabolic type, and hence, it has it has only one family of characteristic lines. If we rewrite the equation in the form

$$ku_{xx} + \cdots\ldots\ldots\ldots = 0,$$

Where the dots stand for the lower order terms, then you can see that the coefficients of the leading order terms are

$$A = k, B = C = 0$$

The slope of the characteristics lines will be given by,

$$\frac{dt}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = 0$$

Consequently, the single family of characteristics lines will be given by

$$t = c$$

These characteristic lines are not very helpful, since they are parallel to the x axis. Thus, one cannot trace points in the xt plane along the characteristics to the x axis, along which the initial data is defined. Notice that there is also no way to factor the heat equation into first order equations, either, so the methods used for the wave equation do not shed any light on the solutions of the heat equation. Instead, we will study the properties of the heat equation, and use the gained knowledge to devise a way of reducing the heat equation to an ODE, as we have done for every PDE, as we have solved so far.

3 Analytical Solution of Heat Equation

3.1 Method of Characteristics

In mathematics, the method of characteristics is a technique for solving partial differential equations. Typically, it applies to first-order equations, although more generally the method of characteristics is valid for any hyperbolic partial differential equation. The method is to reduce a partial differential equation to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hyper surface. The equations in the problems we have investigated so far are all linear and the terms containing the unknown function and its derivatives have constants coefficients. The only exception is the type of problem when we need to make use of polar coordinates, but in such problems the polar radius is present in some of the coefficients in a very specific way, which does not disturb the solution scheme.

3.2 Solution of Heat Equation

Let us now consider the solution of the 1-dimensional heat equation

$$\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Subject to non-homogeneous boundary conditions

$$T(0,t) = T_1, T(L,t) = T_2, T(x,0) = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Which might correspond to a situation where a long rod with an initial temperature distribution f(x) has its two ends inserted into different heat baths that are maintained at different temperatures.

Since we expect that eventually as $t \to \infty$ the rod will eventually reach a steady state temperature distribution that is independent of time, we shall suppose that if

$$for\ t\ sufficiently\ large\ T(x,t) \approx T_{ss}(x)$$

Where $T_{ss}(x)$ is the (as yet undetermined) final steady state temperature distribution. Since even for large t, T(x,t) must still satisfy (i), (ii) , we have for sufficiently large t

$$0 = \frac{\partial T_{ss}}{\partial t} - \alpha^2 \frac{\partial^2 T_{ss}}{\partial x^2} \ => \frac{\partial^2 T_{ss}}{\partial x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

$$and\ T_{ss}(0) = T_1, T_{ss}(L) = T_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (iv)$$

$$The\ differential\ equation\ \frac{\partial^2 T_{ss}}{\partial x^2} = 0\ implies\ T_{ss}\ is\ a\ linear\ function\ of\ x,$$

$$T_{ss}(x) = Ax + B$$

And the boundary conditions (iv) require the constants A and B to be

$$B = T_1 \ \ and\ A = \frac{T_2 - T_1}{L}$$

$$Thus\ \ T_{ss}(x) = \frac{T_2 - T_1}{L} x + T_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Let us now define an auxiliary function $\tau\,(x,t)$ by

$$T(x,t) = T_{ss}(x) + \tau\,(x,t) \ldots\ldots\ldots\ldots\ldots\ldots . (vi)$$

Evidently, $\tau(x,t)$ represents the discrepancy between the actual solution and the final steady state solution. Plugging the righthand side of (vi) into equations (i) and (ii) we find (noting again $\frac{d^2 T_{ss}}{dx^2} = 0 = \frac{\partial T_{ss}}{\partial t}$)

$$\frac{\partial \tau}{\partial t} - \alpha^2 \frac{\partial^2 \tau}{\partial x^2} = 0$$

$$And\ T_1 = T(0,t) = T_{ss}(0) + \tau(0,t) = T_1 + \tau(0,t)$$

$$=> \tau(0,t) = 0$$

$$T_2 = T(L,t) = T_{ss}(L) + \tau(L,t) = T_2 + \tau(L,t)$$

$$=> \tau(L,t) = 0$$

$$f(x) = T(x,0) = T_{ss}(x) + \tau(x,0) = \frac{T_2 - T_1}{L} x + T_1 + \tau(x,0)$$

$$=> \tau(x,0) = f(x) - \frac{T_2 - T_1}{L} x - T_1$$

$$Thus\ \tau(x,t)\ satisfies\ \ \frac{\partial \tau}{\partial t} - \alpha^2 \frac{\partial^2 \tau}{\partial x^2} = 0,$$

$$\tau(0,t) = 0, \tau(L,t) = 0, \tau(x,0) = F(x)$$

$$Where\ F(x) = f(x) - \frac{T_2 - T_1}{L} x - T_1$$

In other words, a PDE / BVP of the form (v), (vi), (vii). We can thus conclude from the results of the last section that

$$\tau(x,t) = \sum_{n=0}^{\infty} c_n\, e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin(\frac{n\pi}{L} x)\,,$$

$$where\ c_n = \frac{2}{L} \int_0^L F(x) \sin(\frac{n\pi}{L} x)\, dx$$

Hence, the solution of equations (i) and (ii) is

$$T(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum_{n=0}^{\infty} c_n\, e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin(\frac{n\pi}{L} x)$$

$$where\ c_n = \frac{2}{L} \int_0^L \left(f(x) - \frac{T_2 - T_1}{L} x - T_1\right) \sin(\frac{n\pi}{L} x)\, dx$$

3.3 Problem Solve

(a) Problem solve by separation of variable

Consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; 0 < x < 1, u(0,t) = u(1,t) = 0, u(x,0) = \sin x$$

Solution: $Given\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$$u(0,t) = u(1,t) = 0, u(x,0) = \sin x \ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (ii)$$

Let,

$u(x,t) =$

$X(x)T(t) \ldots\ldots\ldots .. (iii)$ $[Where\ X\ is\ a\ function\ of\ x\ and\ T\ is\ a\ func$

$$\frac{\partial u}{\partial t} = XT' \ \ and\ \ \frac{\partial u}{\partial x} = X'T \ => \frac{\partial^2 u}{\partial x^2} = X''T$$

From equation (i) we get,

$$XT' = X''T => \frac{T'}{T} = \frac{X''}{X}$$

Since X is a function of x and T is a function of t and they are equal. So they must be equal to constant.

$$\therefore \frac{T'}{T} = \frac{X''}{X} = -\lambda^2 \quad (Say)$$

$$\therefore \frac{X''}{X} = -\lambda^2 \quad and\ \frac{T'}{T} = -\lambda^2$$

$$=> X'' + \lambda^2 X = 0 \dots\dots\dots\dots\dots\dots\dots (iv), \quad T'' + \lambda^2 T = 0 \dots\dots\dots\dots\dots\dots\dots (v)$$

Solution of (iv) is,

$$X(x) = c_1 \cos\lambda x + c_2 \sin\lambda x$$

Solution of (v) is,

$$T(t) = c_3 e^{-\lambda^2 t}$$

From equation (iii) we get,

$$u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x).c_3 e^{-\lambda^2 t}$$

$$=> u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\lambda^2 t} \dots\dots\dots\dots\dots.(vi) \quad [where\ A = c_1 c_3\ and\ B = c_2 c_3]$$

Now, applying initial condition we get,

$$u(0,t) = (A.1 + B.0)e^{-\lambda^2 t}$$

$$=> 0 = Ae^{-\lambda^2 t}$$

$$\therefore A = 0 \quad \left[since\ e^{-\lambda^2 t} \neq 0\right]$$

Again,

$$u(1,t) = (A\cos\lambda + B\sin\lambda)e^{-\lambda^2 t}$$

$$=> 0 = B\sin\lambda\, e^{-\lambda^2 t}$$

$$=> \sin\lambda = 0 = \sin n\pi$$

$$\therefore \lambda = n\pi$$

Now putting the value of λ and A in (vi) we get,

$$u(x,t) = e^{-n^2\pi^2 t}.B\sin n\pi x \dots\dots\dots\dots\dots\dots\dots\dots (vii)$$

$$=> u(x,0) = B\sin n\pi x$$

$$=> \sin x = B\sin n\pi x$$

It is possible if $B = 1\ and\ n = \frac{1}{\pi}$

Now from equation (vii) we get,

$$u(x,t) = e^{-t}\sin x$$

Which is the required solution.

(b) Problem solve by Fourier Transformation

Consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; 0 < x < 1, u(0,t) = u(1,t) = 0, u(x,0) = \sin x$$

Solution: Given that,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \dots\dots\dots\dots\dots\dots\dots (i)$$

Taking finite Fourier Sine Transformation, we get,

$$\int_0^1 \frac{\partial u}{\partial t}\sin\frac{n\pi x}{1}dx = \int_0^1 \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{1}dx \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(ii)$$

$$Let, \quad v = v(n,t) = \int_0^1 u(x,t)\sin n\pi x\, dx \dots\dots\dots\dots\dots\dots (iii)$$

$$=> \frac{\partial v}{\partial t} = \int_0^1 \frac{\partial u}{\partial t}\sin n\pi x\, dx = \int_0^1 \frac{\partial^2 u}{\partial x^2}\sin n\pi x\, dx$$

$$=> \frac{\partial v}{\partial t} = \left[\frac{\partial u}{\partial t}\sin n\pi x\right]_0^1 - \int_0^1 n\pi\cos n\pi x.\frac{\partial u}{\partial t}dx$$

$$= 0 - n\pi\int_0^1 \frac{\partial u}{\partial t}\cos n\pi x\, dx$$

$$=> \frac{\partial u}{\partial t} = -n\pi[u(x,t)\cos n\pi x]_0^1 + n\pi\int_0^1 (-n\pi\sin n\pi x)\,u(x,t)dx$$

$$=> \frac{\partial v}{\partial t} = 0 - n^2\pi^2\int_0^1 u(x,t)\sin n\pi x\, dx \qquad [\because U(0,t) = U(1,t) = 0]$$

$$=> \frac{dv}{dt} = -n^2\pi^2 v \qquad [From\ (iii)]$$

$$=> \frac{dv}{v} = -n^2\pi^2 dt$$

$$=> \ln v = -n^2\pi^2 t + \ln A \quad [Integrating]$$

$$=> v = Ae^{-n^2\pi^2 t} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (iv)$$

$$=> v(n,t) = Ae^{-n^2\pi^2 t}$$

$$=> v(n,0) = A \quad [at\ t = 0]$$

$$=> \int_0^1 u(x,0)\sin n\pi x\, dx = A$$

$$=> \int_0^1 \sin x\sin n\pi x\, dx = A \qquad [\because U(x,0) = \sin x]$$

$$=> A = \frac{1}{2}\int_0^1 2\sin x\sin n\pi x\, dx$$

$$= \frac{1}{2}\int_0^1 [\cos(n\pi - 1)x - \cos(n\pi + 1)x]\, dx$$

$$=> A = \frac{1}{2}\int_0^1 \cos(n\pi - 1)x\, dx - \frac{1}{2}\int_0^1 \cos(n\pi + 1)x\, dx$$

$$=> A = \frac{1}{2}\left[\frac{\cos(n\pi - 1)x}{(n\pi - 1)}\right]_0^1 - \frac{1}{2}\left[\frac{\cos(n\pi + 1)x}{(n\pi + 1)}\right]_0^1$$

$$=> A = \frac{1}{2}\left[\frac{\cos(n\pi - 1)}{(n\pi - 1)} - \frac{\cos(n\pi + 1)}{(n\pi + 1)}\right]$$

$$\therefore v = \frac{1}{2}e^{-n^2\pi^2 t}\left[\frac{\cos(n\pi - 1)}{(n\pi - 1)} - \frac{\cos(n\pi + 1)}{(n\pi + 1)}\right]$$

By inverse Sine Transformation we get,

$$u(x,t) = \frac{2}{1}\sum_{n=1}^{\infty}\frac{1}{2}e^{-n^2\pi^2 t}\left[\frac{\cos(n\pi - 1)}{(n\pi - 1)} - \frac{\cos(n\pi + 1)}{(n\pi + 1)}\right]\sin n\pi x$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\left[\frac{\cos(n\pi - 1)}{(n\pi - 1)} - \frac{\cos(n\pi + 1)}{(n\pi + 1)}\right]\sin n\pi x$$

Which is our required solution.

(b) Problem solve by Laplace Transformation

Consider the initial boundary value problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}; 0<x<1, u(0,t)=u(1,t)=0, u(x,0)=\sin x$$

Solution: Given that

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \dots\dots\dots\dots\dots\dots\dots (i)$$

Now taking Laplace Transformation on both sides we get,

$$L\left\{\frac{\partial u}{\partial t}\right\}=L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$=> su-u(x,0)=\frac{d^2u}{dx^2}$$

$$=> \frac{d^2u}{dx^2}-su=-u(x,0)$$

$$=> \frac{d^2u}{dx^2}-su=-\sin x \dots\dots\dots\dots\dots . (ii)$$

$$Complete\ solution=c_1e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}$$

$$P.I=-\frac{\sin x}{D^2-s}=\frac{\sin x}{s+1}$$

Therefore, the general solution of (ii) is

$$u(x,s)=c_1e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}+\frac{\sin x}{s+1} \dots\dots\dots\dots\dots\dots (iii)$$

Now, $u(0,t)=0$

$$=> L\{u(0,t)\}=0$$
$$=> u(0,s)=0$$

Again, $u(1,t)=0$

$$=> L\{u(1,t)\}=0$$
$$=> u(1,s)=0$$

Now using the 1st condition in (iii) we get,

$$u(0,s)=c_1+c_2$$
$$=> c_1+c_2=0$$
$$=> c_1=-c_2$$

Now using the 2nd condition in (iii) we get,

$$u(1,s)=c_1e^{\sqrt{s}}+c_2e^{-\sqrt{s}}+\frac{\sin 1}{s+1}$$
$$=> c_1e^{\sqrt{s}}-c_1e^{-\sqrt{s}}+0=0$$

$$\left[since\ \sin 1=0.017 \text{ so we can write } \frac{\sin 1}{s+1}=0\right]$$
$$=> c_1\left(e^{\sqrt{s}}-e^{-\sqrt{s}}\right)=0$$
$$=> c_1=0$$

$$\therefore c_1=c_2=0$$

Thus equation (iii) becomes,

$$u(x,s)=\frac{\sin x}{s+1}$$

Taking inverse Laplace Transformation we get,

$$u(x,t)=e^{-t}\sin x$$

Which is the required solution.

4 Experiments and results

We develop a computer program (code) in Matlab programming of scientific computing and implement analytic solution for a heat equation. The main parts of the implementation of our analytic scheme are given as in the following algorithm:

Input: nt and nx are the numbers of grid points of time and space respectively. k and h are the right end points of [0,k] and [0,h].
uo is as a initial condition and ua as a boundary condition.
Output: u(t,x) is the solution matrix.

Step 1: **Initialization:**

```
k=t(2)-t(1);
h=x(2)-x(1);
```

Step 2: **Calculation of Analytic solution:**

```
        for e=2:nt
                for f=2:nx
                        z(e,f)=exp(-t(e))*sin(x(f));
                end
        end
end
surf(t,x,u)
title('Figure of Numerical Scheme');
xlabel('t-axis');

ylabel('x-axis');
```

Step 3: Print u(t,x)
Step 4: Stop

To test the accuracy of the implementation of the analytic scheme, we consider the heat equation. Now we show our results:

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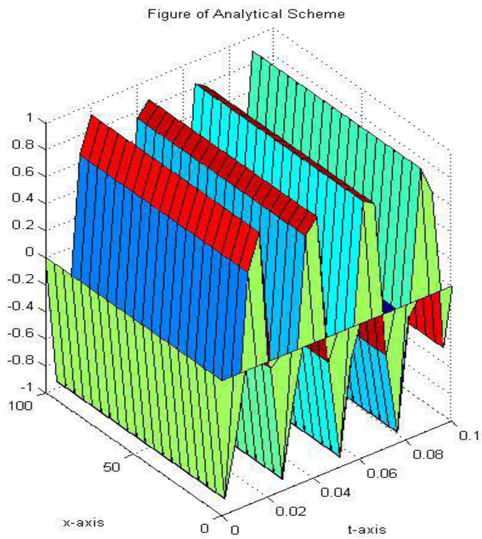

5 Conclusion

In this paper we have considered the second order heat equation. First, we have shown that fundamentals of heat equation, analytical solution by using separation of variable method, Fourier transform method and Laplace transform method. Finally, we show the analytical solution in Matlab computer programming. In future work, we implement the numerical scheme and also compare in heat equation

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